Impact of Random Pointing Errors on CLR Efficiency and Scintillation

Philip Gatt and Scott Shald
Lockheed Martin Coherent Technologies
135 South Taylor Avenue
Louisville, CO 80027

Presented at the
VXI Coherent Laser Radar Conference
June 20-24, 2011

Funded by Lockheed Martin IRAD

Problem Overview/History

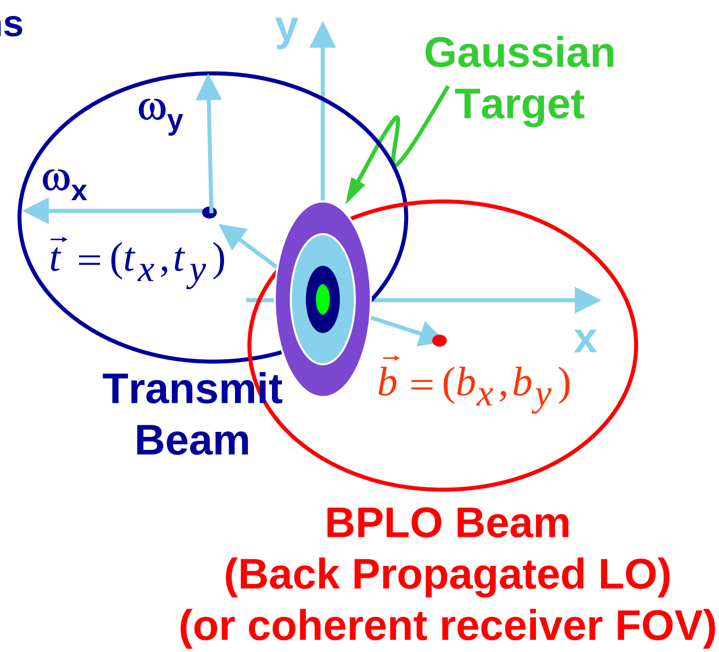
- The goal of this R&D was to develop **analytic expressions** and **"rules of thumb"** for specifying pointing requirements for a coherent laser radar
  - Random pointing errors result in:
    - Reduced average signal power
    - Increased signal variability

- We found very little published work in the literature
  - Takenaka (1978) and Wang (1984) address **fixed offset errors** for **point targets**
  - Yura (1994) considers the impact of transmitter pointing errors on **direct detection** ladars for both **point and extended targets** and assumes a large receiver FOV so that receiver pointing errors are inconsequential
  - Papurt & Shapiro give an **integral expression** for the scenario **with point targets,** but **assume only transmitter pointing errors**
  - Youmans (1997) solved the **coherent ladar** problem for **point targets** with **circular Gaussian beams** assuming **full transmit/receive correlation** and **identical bias**
  - Frehlich (1991) and Andrews (2001) setup similar mathematics for ladar problems associated with the random pointing errors induced by turbulence

Technical Contribution of this Work

- Derived generalized analytic expressions for
  - Efficiency and signal scintillation
  - Special case solutions shown here, leading to "rules of thumb"
  - Generalized expressions in the Proceedings

- Generalized solutions assume
  - Elliptical Gaussian transmit and receiver (BPLO) beams
  - Arbitrary (-1 to 1) correlation between the transmitter and receiver pointing
  - Diffuse elliptical Gaussian target
    - Converges to unresolved point-like targets and uniform extended targets
  - Elliptical Gaussian LOS pointing jitter
    - Correlation between the two axes is ignored
  - Beam, target and jitter ellipse axes are aligned
    - Needed assumption to yield an analytic solution
  - Short waveform integration time compared to LOS jitter time constant
    - "Frozen" motion



Pointing Efficiency and Signal Scintillation: Definitions and General Solutions

$$\eta = \frac{E[P_S(\vec{\mu},\vec{\sigma})]}{E[P_S(0,0)]}$$

- **Efficiency due to static beam offset or misalignment**

$$\eta_o(\vec{t},\vec{b}) = \frac{\int \rho_\pi(\vec{r}) I_t(\vec{r}-\vec{t}) I_b(\vec{r}-\vec{b}) d\vec{r}}{\int \rho_\pi(\vec{r}) I_t(\vec{r}) I_b(\vec{r}) d\vec{r}} \qquad \vec{t} = (t_x, t_y) \text{ and } \vec{b} = (b_x, b_y)$$

$$\eta_o(\vec{t},\vec{b}) = \exp\frac{-2(\omega_{Tx}^2 (t_x - b_x)^2 + \omega_x^2 (t_x^2 + b_x^2))}{\omega_x^2(\omega_x^2 + 2\omega_{Tx}^2)} \exp\frac{-2(\omega_{Ty}^2 (t_y - b_y)^2 + \omega_y^2 (t_y^2 + b_y^2))}{\omega_y^2(\omega_y^2 + 2\omega_{Ty}^2)}$$

- **Jitter plus bias efficiency is found using Conditional Statistics**
  - **Integrate or average the bias efficiency over the distribution of pointing errors**

$$\eta = \iint \eta_o(\vec{t},\vec{b}) f(\vec{t};\vec{b}) d\vec{t}\, d\vec{b}$$

- **Scintillation index is related to the power estimate SNR**

$$\delta_P^2 = \mathrm{var}[P]/\mu_P{}^2 \qquad SNR = 1/\delta_P^2 \le 1 \text{ with speckle}$$

- **Math is simpler in terms of normalized 2nd moments**

$$\gamma_P^2 = \iint \eta_o^2(\vec{t},\vec{b}) f(\vec{t},\vec{b}) d\vec{x}\, d\vec{b}$$

- **Scintillation index is the normalized 2nd moment -1**

$$\delta_P^2 = \gamma_P^2 - 1$$

Circularly Symmetric Pointing Efficiency: Correlated and Uncorrelated Transmit/Receive Cases

- **Gaussian target solution, arbitrary ω_T**

$$\eta|_{\rho=0} = \exp\left[-\frac{2[(\omega^2+4\sigma^2)(|\vec{\mu}_t|^2+|\vec{\mu}_b|^2)+\omega_T^2(\vec{\mu}_t-\vec{\mu}_b)^2]}{(\omega^2+4\sigma^2)[(\omega^2+4\sigma^2)+2\omega_T^2]}\right]\frac{\omega^2}{(\omega^2+4\sigma^2)}\frac{\omega^2+2\omega_T^2}{[(\omega^2+4\sigma^2)+2\omega_T^2]}$$

$$\eta|_{\rho=1} = \exp\left[-\frac{2[\omega^2(|\vec{\mu}_t|^2+|\vec{\mu}_b|^2)+(\omega_T^2+4\sigma^2)(\vec{\mu}_t-\vec{\mu}_b)^2]}{\omega^2(\omega^2+2(\omega_T^2+4\sigma^2))}\right]\frac{\omega^2+2\omega_T^2}{\omega^2+2(\omega_T^2+4\sigma^2)}$$

- **Point target solution, $\omega_T \to 0$**

$$\eta|_{\rho=0} = \exp\left[-\frac{2(|\vec{\mu}_t|^2+|\vec{\mu}_b|^2)}{\omega^2+4\sigma^2}\right]\frac{\omega^4}{(\omega^2+4\sigma^2)^2}$$

$$\eta|_{\rho=1} = \exp\left[-\frac{2[(|\vec{\mu}_t|^2+|\vec{\mu}_b|^2)+4(\sigma^2/\omega^2)(\vec{\mu}_t-\vec{\mu}_b)^2]}{\omega^2+8\sigma^2}\right]\frac{\omega^2}{\omega^2+8\sigma^2}$$

Square Law

Square Law

Youmans, 1997, Eq. 11, with $u_x = u_b$

- **Extended target solution, $\omega_T \to \infty$**

$$\eta|_{\rho=0} = \exp\left[-\frac{(\vec{\mu}_t-\vec{\mu}_b)^2}{\omega^2+4\sigma^2}\right]\frac{\omega^2}{\omega^2+4\sigma^2}$$

Note, $\frac{\omega^2}{\omega^2+4\sigma^2}$ is the ratio of beam areas (unperturbed to perturbed long term average)

$$\eta|_{\rho=1} = \exp\left[-\frac{(\vec{\mu}_t-\vec{\mu}_b)^2}{\omega^2}\right]$$

No dependence on jitter, only bias

Scintillation Index with Speckle Modulation

- Speckle modulation is modeled as a multiplicative process

$$P_S = S\,P$$

  - where S is a unit mean exponentially distributed random variable representing speckle modulation and P is the signal power with beam jitter and bias

- Since the modulation variables are independent, the nth moment of the modulated power is the product of the independent nth moments of S and P.

$$< P_S^n > = < S^n >< P^n >$$

  - Therefore

$$\gamma_{P_S}^2 = \gamma_S^2 \gamma_P^2$$

- For Exponential RVs (speckle) the normalized 2nd moment is 2.
  - Therefore

$$\gamma_{P_S}^2 = 2\gamma_P^2 \quad \text{and} \quad \delta_{P_S}^2 = 2\gamma_P^2 - 1$$

Gaussian target solution, arbitrary ωt

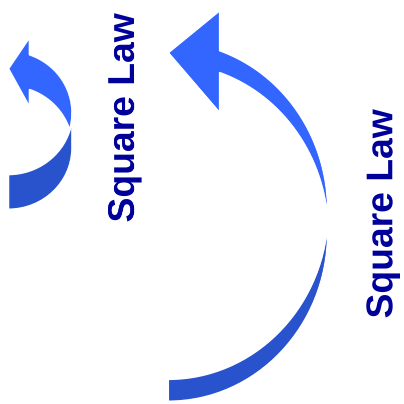
$$\gamma^2_{P|\rho=0} = \exp\left[\frac{4[(\omega^2+4\sigma^2)(|\vec{\mu}_t|^2+|\vec{\mu}_b|^2)+\omega_t^2(\vec{\mu}_t-\vec{\mu}_b)^2]}{(\omega^2+4\sigma^2)(\omega^2+2\omega_t^2+4\sigma^2)} - \frac{4[(\omega^2+8\sigma^2)(|\vec{\mu}_t|^2+|\vec{\mu}_b|^2)+\omega_t^2(\vec{\mu}_t-\vec{\mu}_b)^2]}{(\omega^2+8\sigma^2)(\omega^2+2\omega_t^2+8\sigma^2)}\right]\frac{(\omega^2+4\sigma^2)^2(\omega^2+2\omega_t^2+4\sigma^2)^2}{\omega^2(\omega^2+2\omega_t^2)(\omega^2+8\sigma^2)(\omega^2+2\omega_t^2+8\sigma^2)}$$

$$\gamma^2_{P|\rho=1} = \exp\left[\frac{4[\omega^2(|\vec{\mu}_t|^2+|\vec{\mu}_b|^2)+(\omega_t^2+4\sigma^2)(\vec{\mu}_t-\vec{\mu}_b)^2]}{\omega^2(\omega^2+2(\omega_t^2+4\sigma^2))} - \frac{4[\omega^2(|\vec{\mu}_t|^2+|\vec{\mu}_b|^2)+(\omega_t^2+8\sigma^2)(\vec{\mu}_t-\vec{\mu}_b)^2]}{\omega^2(\omega^2+2(\omega_t^2+8\sigma^2))}\right]\frac{(\omega^2+2(\omega_t^2+4\sigma^2))^2}{(\omega^2+2\omega_t^2)[\omega^2+2(\omega_t^2+8\sigma^2)]}$$

Point target solution, $\omega t \to 0$

$$\gamma^2_{P|\rho=0} = \exp\left[\frac{4(|\vec{\mu}_t|^2+|\vec{\mu}_b|^2)}{(\omega^2+4\sigma^2)} - \frac{4(|\vec{\mu}_t|^2+|\vec{\mu}_b|^2)}{(\omega^2+8\sigma^2)}\right]\frac{(\omega^2+4\sigma^2)^4}{\omega^4(\omega^2+8\sigma^2)^2}$$

$$\gamma^2_{P|\rho=1} = \exp\left[\frac{4[\omega^2(|\vec{\mu}_t|^2+|\vec{\mu}_b|^2)+4\sigma^2(\vec{\mu}_t-\vec{\mu}_b)^2]}{\omega^2(\omega^2+8\sigma^2)} - \frac{4[\omega^2(|\vec{\mu}_t|^2+|\vec{\mu}_b|^2)+8\sigma^2(\vec{\mu}_t-\vec{\mu}_b)^2]}{\omega^2(\omega^2+16\sigma^2)}\right]\frac{(\omega^2+8\sigma^2)^2}{\omega^2(\omega^2+16\sigma^2)}$$



Extended target solution, $\omega t \to 0$

$$\gamma^2_{P|\rho=0} = \exp\left[\frac{2(\vec{\mu}_t-\vec{\mu}_b)^2}{(\omega^2+4\sigma^2)} - \frac{2(\vec{\mu}_t-\vec{\mu}_b)^2}{(\omega^2+8\sigma^2)}\right]\frac{(\omega^2+4\sigma^2)^2}{\omega^2(\omega^2+8\sigma^2)}$$

$$\gamma^2_{P|\rho=1} = 1$$

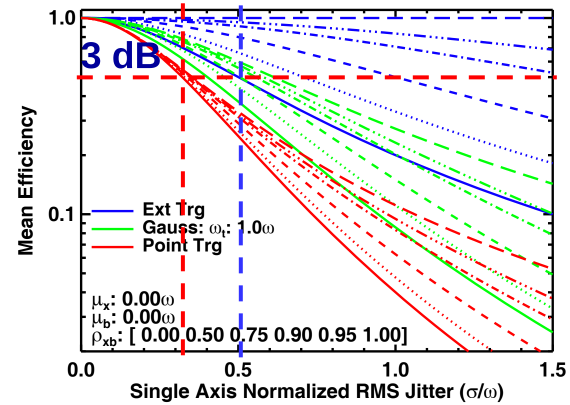
No change in scintillation over speckle modulation

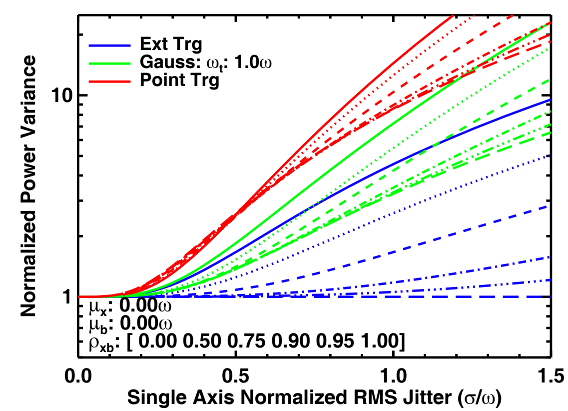
Theoretical Predictions vs. RMS Jitter with Zero Bias

Theoretical predictions parametric in correlation

- Large Targets with $\rho \sim 1$
  - Efficiency ~ 1
  - Normalized variance w/o speckle ~ 0 (noiseless)
  - SNR ~ 1
- Small targets with $\rho \sim 1$
  - Efficiency < 1
  - Normalized variance, w/o speckle, > 0
  - SNR < 1
- When $\rho < 1$
  - Less efficiency
  - Greater scintillation
- For efficiency > 50% with uncorrelated beams
  - $\sigma < \sim 0.50\omega$ with extended targets
  - $\sigma < \sim 0.35\omega$ with point targets
- For better than 90 % pointing efficiency, pointing error should be
  - $\sigma < \sim 0.20\omega$ with extended targets
  - $\sigma < \sim 0.10\omega$ with point targets

zero bias





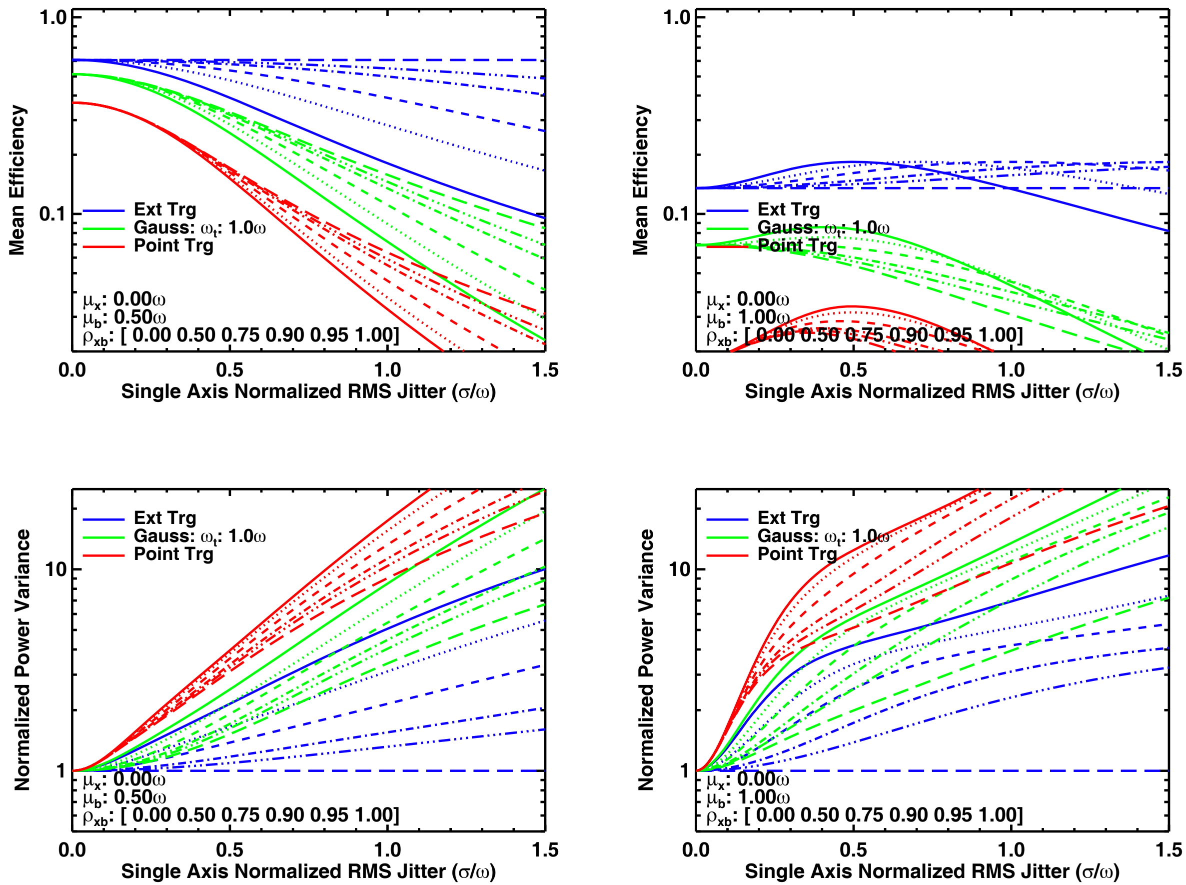
Theoretical Predictions vs. RMS Jitter with BPLO Bias

Theoretical predictions parametric in correlation with BPLO bias and zero transmit bias

- **Bias reduces efficiency and increases variance**
- **Bias should be kept less than a small fraction of a beam radius for < 3 dB loss**
  - $\mu <\sim 0.50\omega$ extended trg
  - $\mu <\sim 0.25\omega$ point trg
- **Uncorrelated beam motion actually improves performance under large bias, when efficiency is already very low by**
  - increasing probability of beam and target overlap

[(0 ,0); (½,½)]ω bias

[(0 ,0); (1,1)]ω bias

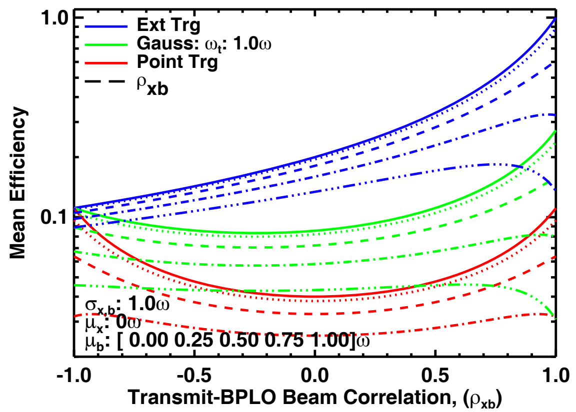


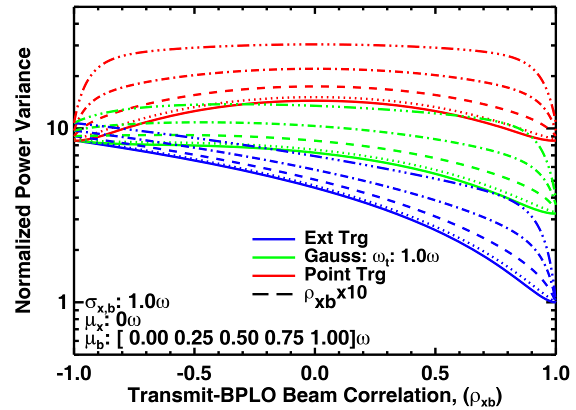
Theoretical Predictions vs. Transmit/Receive Correlation:

- **Efficiency and Normalized Variance Conclusions**
  - **Correlated beams**
    - **Best performance for all targets except with extreme bias**
  - **Uncorrelated beams**
    - **Worst case for point targets**
  - **Anti-correlated beams**
    - **Worst case for large targets**

- **Normalized Variance (Scint+1)**
  - **Inversely related to efficiency**
    - **Better efficiency => less scintillation**

rms jitter, $\sigma = \omega$

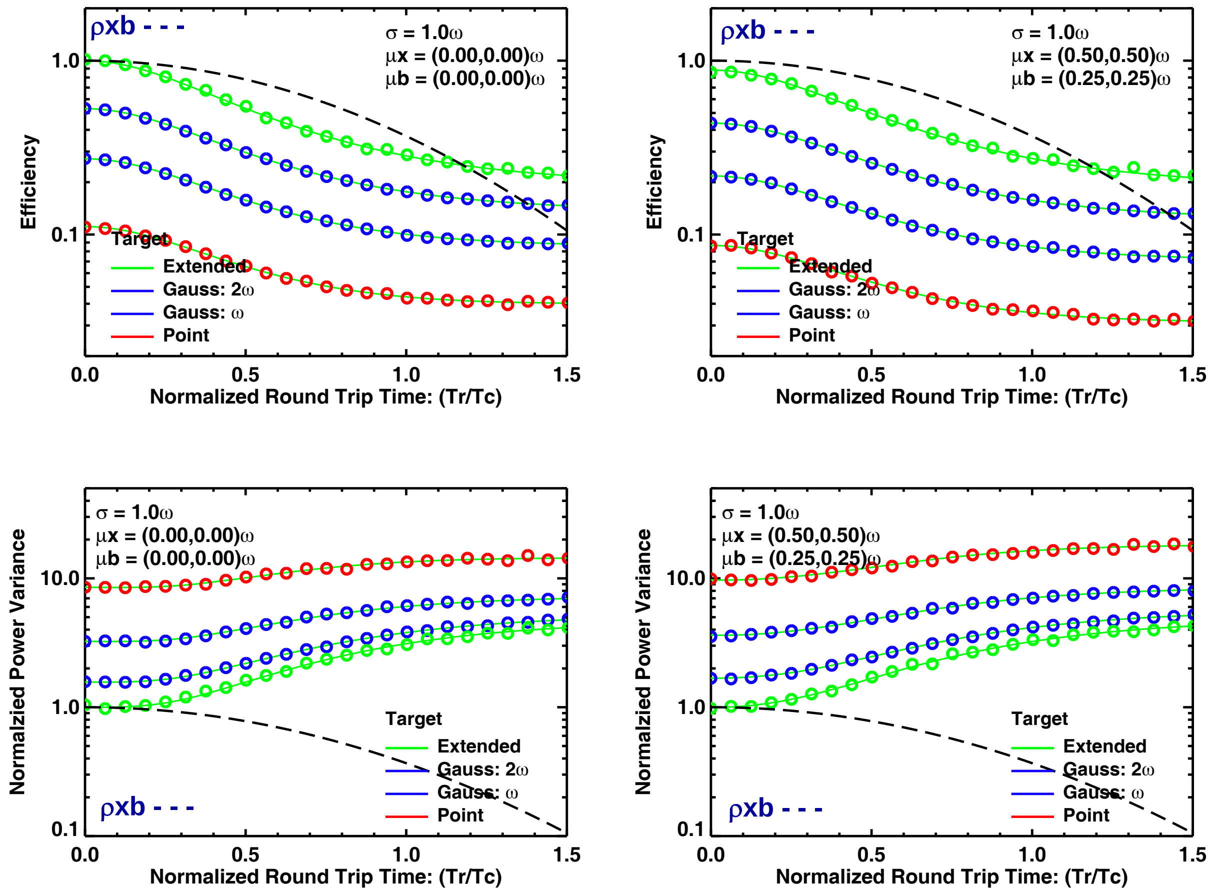




Theoretical Predictions vs. Target Range
Comparing Monte-Carlo Simulation Results to Theoretical Predictions

- **Goal: Develop a numerical experiment tool for evaluating non-Gaussian cases.**
- **Correlation depends on target range and pointing PSD**
  - **Gaussian PSD assumed -> Gaussian ACF with exp-1 time, Tc.**
- **Many (10,000) independent samples**
- **Numerical experiment tool validated by theoretical predictions**
  - **o-o-o -> Experiment**
  - **______ -> Theory**
  - **- - - - -> ρxb**
- **Virtual experiment can be readily modified for**
  - **Truncated beams w/ side-lobes**
  - **Complex target shapes**

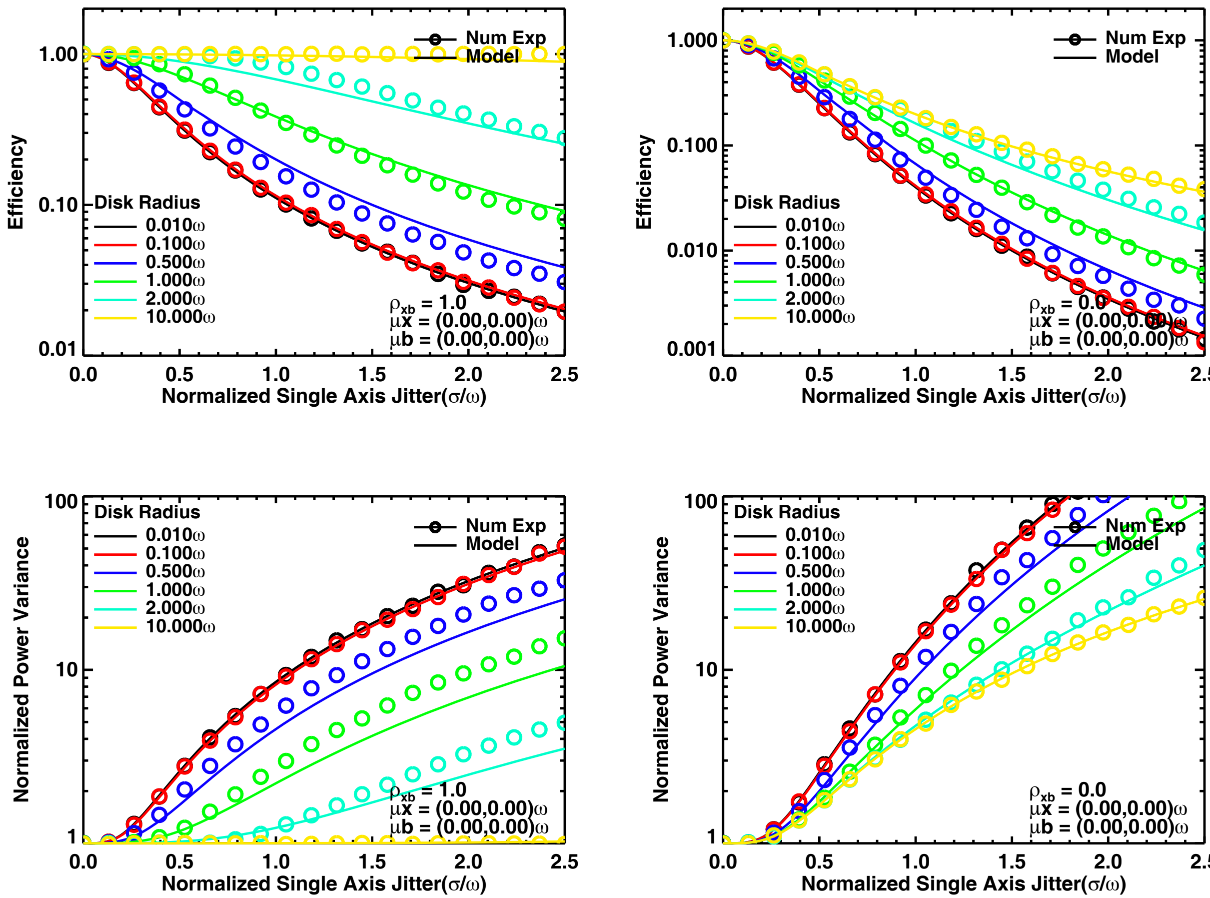
zero bias | **[(½ ,½); (¼,¼)]ω bias**



Modeling Disk Target Performance w/ Gaussian Theory

- **When the target radius is much smaller than the beam radius**
  - $r_t << \omega$
  - Any small enough Gaussian target suffices
- **When $r_t >> \sqrt{(\omega^2+4\sigma^2)}$**
  - Any large enough Gaussian target suffices
- **Model which matches LRCS for small targets**
  - $\omega_t = \sqrt{2}\, r_t$
- **Model shown as solid line vs. numerical experiment, o's**
- **Reasonably good agreement is achieved over spread of target sizes and jitter**

Correlated Beams | **Uncorrelated Beams**



Summary

- This analysis assumed
  - Elliptical Gaussian beams, targets and pointing errors
  - Arbitrary correlation between the transmit and BPLO beam positions

- When the round-trip target time (~ 1 ms/150 km) exceeds the LOS correlation time ($t_c \sim f_c/2\pi$)
  - Transmit and receiver pointing are virtually uncorrelated

- Generalized theories and numerical experiments were developed
  - Theoretical predictions agree with special cases examined by others (e.g., Youmans 1997)
  - Experiment results agreed with theoretical predictions

- For high efficiency and small variability
  - Both bias and jitter should be kept to a small fraction (< ¼ to ½) of a beam radius, ω, depending upon the target to beam size ratio and correlation
  - Smaller targets need better pointing for the same efficiency
  - Consider a 10 cm aperture with 8 cm ($2\omega_o$) matched Gaussian beams at 1 um wavelength
    - ½ angle = $\lambda/\pi\omega_o$ ~ 8 urad
    - Therefore need rms pointing accuracy less than ~ < 2 urad for a 10 cm aperture
      - or < 200 nrad for a 1 m aperture

- General rules of thumb for beam jitter efficiency
  - Infinite target (correlated beams): ~ 1
  - Infinite target (uncorrelated beams): Ratio of un-perturbed beam area to long term beam area
  - Point target (correlated beams): ~ Same loss as correlated beams with infinite target (e.g., beam area ratio)
  - Point target (uncorrelated beams): ~ Square of loss for an extended target with uncorrelated beams
  - Point target (uncorrelated beams): ~ Square of loss for a point target with correlated beams

- Disk target performance is well modeled with a Gaussian target using $\omega_t = \sqrt{2}\, r_t$
  - Same LRCS as small disk target
  - Reasonably good fit to efficiency and normalized power variance

References

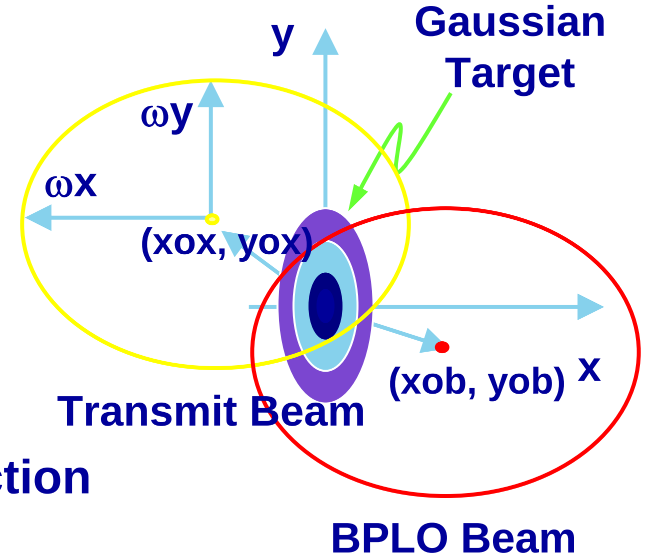
- T. Takenaka, K. Tanaka, and O. Fukumitsu, "Signal-to-Noise ratio in optical heterodyne detection for Gaussian fields," Appl. Opt. 17, 3466 (1978).
- Wang, “Detection efficiency of coherent optical radar,” Appl. Opt. 23, 3421-3427 (1984).
- H. T. Yura, “LADAR detection statistics in the presence of pointing errors,” Appl. Opt. 33, 6482-6498 (1994).
- D. M. Papurt, J. H. Shapiro, S. T. Lau, “Measured turbulence and speckle effects in laser radar target returns,” Proc. SPIE 415, 166-178 (1983).
- D. G. Youmans, R. Robertson, “Modelocked-laser laser radar performance in the detection of TMD and NMD targets”, AIAA/BMDO Technology Readiness Conference, Accession Number ADA329046, (1997).
- Rod G. Frehlich and Michael J. Kavaya, “Coherent laser radar performance for general atmospheric refractive turbulence,” Appl. Opt. 30, 5325-5352 (1991)
- L. C. Andrews, R. L. Phillips, C. Y. Hopen, Laser Beam Scintillation with Application, (SPIE Press, 2001).

Backup

Assumptions (1/2)

- In coherent ladar, the forward propagated (space and time) transmit and back propagated (space and time) LO field distributions interact with the target's complex scattering function to form the complex heterodyne signal
  - BPLO formalism provides insight and simplifies mathematics
- Matched Gaussian transmit and BPLO beams
  - exp[-2] widths: ωx, ωy

$$I_t(x,y) = \frac{\sqrt{2/\pi}}{\omega_x}\exp[\frac{-2(x-t_x)^2}{\omega_x{}^2}]\frac{\sqrt{2/\pi}}{\omega_y}\exp[\frac{-2(y-t_y)^2}{\omega_y{}^2}]$$



- Elliptical Diffuse Gaussian target reflectance function
  - Centered on the origin
  - Exp[-2] widths: ωtx, ωty

$$\rho_\pi(x,y) = \rho_{\pi_o}\exp[\frac{-2x^2}{\omega_{Tx}{}^2}]\exp[\frac{-2y^2}{\omega_{Ty}{}^2}]$$

  - Covers span from unresolved point-like targets to uniform extended targets

Assumptions (2/2)

- **Elliptical Gaussian LOS jitter with bias (μx, μy) and rms (σx, σy) in each axis**
  - **For Gaussian beams the average beam with Gaussian jitter remains Gaussian with exp[-2] width**

$$\omega_{x\sigma}^2 = \omega_x{}^2 + 4\sigma_x^2 \qquad \omega_{y\sigma}^2 = \omega_y{}^2 + 4\sigma_y^2$$

  - **rms pointing for both beams are equal but unique for each axis (i.e., $\sigma tx = \sigma bx$ and $\sigma ty = \sigma by$)**

$$f(t_x, t_y; b_x, b_y) = \frac{1}{2\pi\sigma^2\sqrt{(1-\rho^2)}} \times \frac{\exp[-(t_x-\mu_{tx})^2 + (b_x-\mu_{bx})^2 - 2\rho(t_x-\mu_{tx})(b_x-\mu_{bx}))]}{2\sigma^2(1-\rho^2)}$$

$$\times \frac{\exp[-(t_y-\mu_{ty})^2 + (b_y-\mu_{by})^2 - 2\rho(t_y-\mu_{ty})(b_y-\mu_{by}))]}{2\sigma^2(1-\rho^2)}$$

  - **ρx, ρy are the correlation coefficients between the transmit and BPLO positions in the x and y directions**
  - **Correlation between the two axes is ignored**

- **Jitter, beam and target ellipse axes are aligned**
  - **Results in a x-y separable problem and the total efficiency is the product of the two orthogonal components**

- **Short waveform integration time compared to LOS jitter time constant**

Transmit and BPLO Beam Position Correlation

- **Correlation between the two beam positions will depend on the ladar geometry and the round-trip time delay**
  - **For monostatic systems**
    - **At short range the two beams could be fully correlated**
    - **Correlation decreases as range increases**
  - **For bistatic systems**
    - **The cross correlation function is needed to specify beam correlation**
    - **One might expect the two beams to be uncorrelated at all ranges due to independent random motion**
    - **However their ought to be a component of the motion which is common to both beams (i.e., platform motion) and this term will depend on the time delay**
- **The LOS pointing Power Spectral Density (PSD), for a single axis, is sufficient to describe the correlation for that axis (x or y)**
  - **The time-dependent correlation coefficient, ρ, is equal to normalized (to unit peak) LOS autocorrelation function (ACF)**
  - **The autocorrelation function is the Fourier transform of the PSD**
    - **Gauss PSD $\rightarrow$ Gauss ACF**
    - **Delta Function PSD $\rightarrow$ Sinusoidal ACF**

Effective Average Signal Power

- **Calculating the "effective" average signal power over an ensemble of diffuse targets is an incoherent process resulting in an overlap integral of the transmit and BPLO irradiance distributions, Ix and Ib, with the target power reflectance function, $\rho\pi$**

$$P_s(t) = \eta P_x(t-t_r) T^2 \lambda^2 \iint \rho_\pi(x,y) I_x(x,y;t-t_r) I_b(x,y;t)\,dxdy$$

- **On Axis point target with cross-section σ**

$$P_s(t) = \eta T^2 P_x(t-t_r)\frac{\lambda^2}{\pi\omega_x\omega_y}\frac{\sigma/\pi}{\pi\omega_x\omega_y}$$

- **Uniform extended target with reflectance, $\rho\pi$with radii (ωx,ωy)**

$$P_s(t) = \eta T^2 P_x(t-t_r)\frac{\lambda^2}{\pi\omega_x\omega_y}\rho_\pi$$

- **Elliptical Gaussian target with radius ωt**

$$P_s(t) = \eta T^2 P_x(t-t_r)\frac{\lambda^2}{\pi\omega_x\omega_y}\rho_\pi\sqrt{\frac{2\omega_{tx}^2}{(\omega_x^2+2\omega_{tx}^2)}}\sqrt{\frac{2\omega_{ty}^2}{(\omega_y^2+2\omega_{ty}^2)}}$$

- **Uniform circular disk target with radius rt**

$$P_s(t) = \eta T^2 P_x(t-t_r)\frac{\lambda^2}{\pi\omega^2}\rho_\pi[1-\exp(-4r_t^2/\omega^2)]$$

Pointing Efficiency: Gaussian Targets, Beams and Motion

- **x-y separable and also separable into a bias and jitter term**

$$\eta = \eta_x \eta_y \quad \eta_x = \eta_{\mu x}\eta_{\sigma x} \quad \eta_y = \eta_{\mu y}\eta_{\sigma y}$$

  - **Although jitter affects the bias term the bias efficiency → 1 when bias is zero**

$$\eta_{\mu|_{\mu=0}} = 1 \qquad \eta_{\sigma|_{\sigma=0}} = 1$$

- **Consider only one axis at a time**
  - **Gaussian target solution, arbitrary ω_t**

$$\eta_{\mu_{x,y}} = \exp\left[-\frac{2[\omega^2(\mu_t^2+\mu_b^2)+4\sigma^2(\mu_t^2+\mu_b^2-2\rho\mu_t\mu_b)+\omega_t^2(\mu_t-\mu_b)^2]}{\omega^4+16\sigma^4(1-\rho^2)+8\omega^2\sigma^2+2\omega_T^2[\omega^2+4\sigma^2(1-\rho)]}\right]$$

$$\eta_{\sigma_{x,y}} = \sqrt{\frac{\omega^4+2\omega^2\omega_T^2}{\omega^4+16\sigma^4(1-\rho^2)+8\omega^2\sigma^2+2\omega_T^2[\omega^2+4\sigma^2(1-\rho)]}}$$

  - **Point target solution, $\omega_t \to 0$**

$$\eta_{\mu_{x,y}} = \exp\left[-\frac{2[\omega^2(\mu_x^2+\mu_b^2)+4\sigma^2(\mu_x^2+\mu_b^2-2\rho\mu_x\mu_b)]}{\omega^4+16\sigma^4(1-\rho^2)+8\omega^2\sigma^2}\right] \qquad \eta_{\sigma_{x,y}} = \frac{\omega^2}{\sqrt{\omega^4+16\sigma^4(1-\rho^2)+8\omega^2\sigma^2}}$$

  - **Extended target solution, $\omega_t \to \infty$**

$$\eta_{\mu_{x,y}} = \exp\left[-\frac{(\mu_t-\mu_b)^2}{[\omega^2+4\sigma^2(1-\rho)]}\right] \qquad \eta_{\sigma_{x,y}} = \frac{\omega}{\sqrt{\omega^2+4\sigma^2(1-\rho)}}$$

Normalized Power Variance (a.k.a. Scintillation Index): Definition and General Solution

- **Normalized signal power variance, δ, gives insight to signal power fluctuations about its mean**

$$\delta_P^2 = \text{var}[P]/\mu_P{}^2 = \gamma_P^2 - 1$$

- **Inverse δ^2 is the pointing-limited power estimate SNR**
  - **Much like Shapiro's Image SNR**

$$SNR = 1/\delta_P^2 \le 1 \text{ with speckle}$$

- **Where γ^2 is the signal power normalized 2nd moment which is also equal to the CNR and/or efficiency normalized 2nd moments**

$$\gamma_P^2 = \frac{< P^2(\mu,\sigma) >}{< P(\mu,\sigma) >^2} = \frac{< \eta^2(\mu,\sigma) >}{< \eta(\mu,\sigma) >^2}$$

- **Conditional statistics provide 2nd movement, given the efficiency due to a pointing error and the distribution of errors**

$$\gamma_P^2 = \frac{\iint \eta^2(\vec{x},\vec{b})\big|_{\sigma=0} f(\vec{x},\vec{b})\, d\vec{x}\, d\vec{b}}{\eta(0,0)^2}$$

Norm 2nd Moment: Gaussian Targets, Beams and Motion

- **x-y separable and also separable into a bias and jitter term**

$$\gamma^2 = \gamma_x^2 \gamma_y^2 \qquad \gamma_x^2 = \gamma_{\mu x}^2 \gamma_{\sigma x}^2 \qquad \gamma_y^2 = \gamma_{\mu y}^2 \gamma_{\sigma y}^2$$

- **Consider only one axis at a time**
  - **Gaussian target solution, arbitrary ω_t**

$$\gamma_{\mu x,y}^2 = \exp\left[-\frac{4[\omega_T^2(\mu_t-\mu_b)^2+\omega^2(\mu_t^2+\mu_b^2)+8\sigma^2(\mu_t^2+\mu_b^2-2\rho\mu_t\mu_b)]}{[\omega^2+8\sigma^2(1-\rho)][\omega^2+2\omega T_t^2+8\sigma^2(1+\rho)]}\right] \times \exp\left[\frac{4[\omega_T^2(\mu_t-\mu_b)^2+\omega^2(\mu_t^2+\mu_b^2)+4\sigma^2(\mu_t^2+\mu_b^2-2\rho\mu_t\mu_b)]}{[\omega^2+4\sigma^2(1-\rho)][\omega^2+2\omega_T^2+4\sigma^2(1+\rho)]}\right]$$

$$\gamma_{\sigma x,y}^2 = \frac{[\omega^2+4\sigma^2(1-\rho)][\omega^2+2\omega_T^2+4\sigma^2(1+\rho)]}{\sqrt{[\omega^2+8\sigma^2(1-\rho)][\omega^2+2\omega_T^2+8\sigma^2(1+\rho)]}\sqrt{\omega^4+2\omega^2\omega_T^2}}$$

  - **Point target solution, $\omega_t \to 0$**

$$\gamma_{\mu x,y}^2 = \exp\left[-\frac{4[\omega^2(\mu_t^2+\mu_b^2)+8\sigma^2(\mu_t^2+\mu_b^2-2\rho\mu_t\mu_b)]}{[\omega^2+8\sigma^2(1-\rho)][\omega^2+8\sigma^2(1+\rho)]}\right] \times \exp\left[\frac{4[\omega^2(\mu_t^2+\mu_b^2)+4\sigma^2(\mu_t^2+\mu_b^2-2\rho\mu_t\mu_b)]}{[\omega^2+4\sigma^2(1-\rho)][\omega^2+4\sigma^2(1+\rho)]}\right]$$

$$\gamma_{\sigma x,y}^2 = \frac{[\omega^2+4\sigma^2(1-\rho)][\omega^2+4\sigma^2(1+\rho)]}{\sqrt{[\omega^2+8\sigma^2(1-\rho)][\omega^2+8\sigma^2(1+\rho)]}\,\omega^2}$$

  - **Extended target solution, $\omega_t \to \infty$**

$$\gamma_{\mu x,y}^2 = \exp\left[-\frac{2[(\mu_t-\mu_b)^2]}{[\omega^2+8\sigma^2(1-\rho)]}\right] \times \exp\left[\frac{2[(\mu_t-\mu_b)^2]}{[\omega^2+4\sigma^2(1-\rho)]}\right] \qquad \gamma_{\sigma x,y}^2 = \frac{[\omega^2+4\sigma^2(1-\rho)]}{\omega\sqrt{[\omega^2+8\sigma^2(1-\rho)]}}$$

Numerical Experiment Description

- **Approach**
  - Matched circular Gaussian beam fields were overlapped and integrated with diffuse (random) targets (Gaussian, point, extended and disk).
  - Correlated random beam motion was applied to the transmit and BPLO beams
  - Overlap integral of the fields were evaluated to generate a random realization of the heterodyne signal complex amplitude
  - Many realizations were employed to yield sufficiently accurate signal power statistics

- **Targets**
  - Point: Specular (constant) and diffuse (random)
    - Specular compared to diffuse to verify normalized variance theory
  - Gaussian, Infinite and disk: Diffuse only
    - Diffuse targets comprised of > 1000 scattering centers with circular complex Gaussian statistics and were randomized for each trail (random) beam position
    - Specular infinite targets could be simulated readily, but not currently of much interest

- **Target Round Trip-Time Experiments**
  - Gaussian LOS pointing PSD with BPLO lagging behind the transmit beam

Outline

- Assumptions/Problem Geometry
- Efficiency and Normalized Power Variance (aka Scintillation Index)
  - Generalized theoretical expressions
  - Circularly symmetric beam/target simplified expressions
  - Example predictions
- Numerical Experiments
  - Description
  - Results compared to predictions
  - Modeling disk target performance with a Gaussian target
- Summary and Conclusions