Statistics of the Doppler Lidar Signal in the Turbulent Atmosphere

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1. Introduction

For coherent detection of optical fields scattered by atmospheric particles the Doppler lidar signal is the sum of a large number of random variables. So in terms of statistics three questions are of interest both from the purely theoretical point of view and from the practical one. What kind of the Doppler lidar signal statistics in the turbulent atmosphere? Can we use the central limit theorem in the analysis of the Doppler lidar signal statistics? What is the main reason for the difference of the Doppler lidar signal statistics from the Gaussian one?

2. Doppler Lidar Signal

In the single scattering approximation the aerosol signal can be written in this form [1]

$$j_s(t) = \sum_{m=1}^{N_p} A_m P(t, R_m) e^{2ikR_m(t)}, \quad (1)$$

$$R_m(t) = R_m + u_r(R_m, \phi, \theta)(t - t_0), \quad (2)$$

where $R_m(t) = \mathbf{r}_m(t)\mathbf{n}$ is vector projection of $\mathbf{r}_m(t)$ to the sensing direction $\mathbf{n}$, $R_m = R_m(t_0)$, $\mathbf{r}_m(t)$ is coordinate of m'th particle, $u_r(R_m, \phi, \theta) = \mathbf{u}(R_m, \phi, \theta)\mathbf{n}$ is radial wind velocity at the point $\mathbf{r}_m = \mathbf{r}_m(t_0) = \{R_m, \phi, \theta\}$, $\mathbf{u}(R_m, \phi, \theta)$ is wind velocity field, A_m is scattering amplitude m'th particle, N_p is number of particles, $P(t, R_m)$ is the Doppler lidar pattern.

3. Atmospheric Models

We assume that coordinates of particles, wind velocity and number of particles are independent in statistic terms. Particles are distributed uniformly and independently in scattering volume at initial time. The law of fluctuations of particles number is Poisson distribution, and a distribution law of velocity obeys a law of normal probability distribution [2]. Therefore radial wind velocity fluctuations $u_r'(R_m, \phi, \theta)$ and difference of radial wind velocity fluctuations $u_r'(R_m, \phi, \theta) - u_r'(R_n, \phi, \theta)$ obeys a law of normal probability distribution too:

$$W_1\left(u_r'(R_m, \phi, \theta)\right) = \frac{1}{\sqrt{2\pi\left\langle u_r'^2(R_m, \phi, \theta)\right\rangle}} \exp\left\{-\frac{u_r'^2(R_m, \phi, \theta)}{2\left\langle u_r'^2(R_m, \phi, \theta)\right\rangle}\right\}, \quad (3)$$

$$W_2\left(u_r'(R_m, \phi, \theta) - u_r'(R_n, \phi, \theta)\right) =$$

$$= \frac{1}{\sqrt{2\pi\left\langle\left\{u_r'(R_m, \phi, \theta) - u_r'(R_n, \phi, \theta)\right\}^2\right\rangle}} \exp\left\{-\frac{\left\{u_r'(R_m, \phi, \theta) - u_r'(R_n, \phi, \theta)\right\}^2}{2\left\langle\left\{u_r'(R_m, \phi, \theta) - u_r'(R_n, \phi, \theta)\right\}^2\right\rangle}\right\}, \quad (4)$$

$$\left\langle u_r'^2(\mathbf{r}_m)\right\rangle = \sum_{i,j=1}^{3} K_{ij}(\mathbf{r}_m,\mathbf{r}_m) n_i n_j \ , \tag{5}$$

$$\left\langle \left\{ u_r'(\mathbf{r}_m) - u_r'(\mathbf{r}_n) \right\}^2 \right\rangle = \sum_{i,j=1}^{3} D_{ij}(\mathbf{r}_m,\mathbf{r}_n) n_i n_j \ . \tag{6}$$

where $K_{ij}(\mathbf{r}_m,\mathbf{r}_n)$ is correlation tensor, $D_{ij}(\mathbf{r}_m - \mathbf{r}_n)$ is structural tensor of fluctuations of wind velocity field. For homogeneous and isotropic turbulence and for exponential model [2] correlation tensor of fluctuations of wind velocity field has the following form

$$K_{ij}(r) = K(r)\delta_{ij} + \frac{1}{2} r \frac{dK(r)}{dr}\left(\delta_{ij} - \frac{r_i r_j}{r^2}\right)$$

$$K(r) = \frac{2}{3} e \times \exp(-r/l), \tag{7}$$

where e and l are turbulent kinetic energy and scale of turbulence respectively.

For homogeneous and isotropic turbulence mean radial wind velocity, variance and structural function of radial wind velocity fluctuations are defined by

$$\langle u_r(R_m,\phi,\theta)\rangle = u_r \ , \tag{8}$$

$$\left\langle u_r'^2(R_m,\phi,\theta)\right\rangle = \frac{2}{3} e \ , \tag{9}$$

$$\left\langle \left(u_r'(R_m,\phi,\theta) - u_r'(R_n,\phi,\theta)\right)^2 \right\rangle = = \frac{4}{3} e \left\{ 1 - K\left(\left|R_m - R_n\right|\right)\right\}. \tag{10}$$

4. Statistical Characteristics of Signal

We assume that there are a large number of particles in scattering volume and they are homogeneous ($A_m = A$). For $N_p \gg 1$ signal correlation functions of second and fourth orders were obtained by averaging of random coordinates, velocity and number of particles according to the distribution law and using Eq. (1) – (6). These functions have the following forms

$$\left\langle j_s\left(t+\frac{\tau}{2}\right) j_s^*\left(t-\frac{\tau}{2}\right)\right\rangle = = S \exp\left\{-\frac{\tau^2}{4\tau_0^2}\right\} \int dR_m \left| p\left(t - 2\frac{R_m}{c}\right)\right|^2 e^{2ik\langle u_r(R_m,\phi,\theta)\rangle\tau - 2k^2\left\langle u_r'^2(R_m,\phi,\theta)\right\rangle\tau^2} \ , \tag{11}$$

$$\left\langle \left| j_s\left(t+\frac{\tau}{2}\right)\right|^2 \left| j_s\left(t-\frac{\tau}{2}\right)\right|^2 \right\rangle = \left\langle \left| j_s\left(t+\frac{\tau}{2}\right)\right|^2 \right\rangle \left\langle \left| j_s\left(t-\frac{\tau}{2}\right)\right|^2 \right\rangle + S^2 \exp\left\{-\frac{\tau^2}{2\tau_0^2}\right\} \times \times \int dR_m dR_n \left| p\left(t-2\frac{R_m}{c}\right)\right|^2 \left| p\left(t-2\frac{R_n}{c}\right)\right|^2 e^{-2k^2\left\langle \left(u_r'(R_m,\phi,\theta) - u_r'(R_n,\phi,\theta)\right)^2\right\rangle \tau^2} \ , \tag{12}$$

where $2\tau_0$ is e^{-1} pulse duration, S is signal magnitude, $p(t)$ is normalized Doppler lidar patterns.

5. Homogeneous and Isotropic Turbulence

Let us consider the non-Gaussian characteristics of the Doppler lidar signal in case of homogeneous and isotropic turbulence.

For this turbulent model the signal correlation functions of second and fourth orders become

$$\left\langle j_s\left(t+\frac{\tau}{2}\right) j_s^*\left(t-\frac{\tau}{2}\right)\right\rangle = S\exp\left\{-\frac{\tau^2}{4\tau_0^2}-\frac{4}{3}k^2 e\tau^2+2iku_r\tau\right\}, \quad (13)$$

$$\left\langle \left|j_s\left(t+\frac{\tau}{2}\right)\right|^2 \left|j_s\left(t-\frac{\tau}{2}\right)\right|^2\right\rangle = \left\langle \left|j_s\left(t+\frac{\tau}{2}\right)\right|^2\right\rangle\left\langle \left|j_s\left(t-\frac{\tau}{2}\right)\right|^2\right\rangle + S^2\exp\left\{-\frac{\tau^2}{2\tau_0^2}-\frac{8}{3}k^2 e\tau^2\right\} NG. \quad (14)$$

$$NG = \frac{1}{\sqrt{2\pi}d_{1/2}}\int dr \exp\left\{-\frac{r^2}{2d_{1/2}^2}-4k^2 K(r)\tau^2\right\} \quad (15)$$

where $2d_{1/2} = c\tau_0$. It follows from the Eqs. (13) – (15) that when $NG \neq 1$ the correlation function of the fourth order is not factorable according to the law of Gaussian statistics. Therefore the signal statistics is non-Gaussian.

Eqs. (13) – (15) show that the statistics of the signal is non-Gaussian if $K(r) \neq 0$. The function $K(r)$ describes the correlation of wind velocity fluctuations in the scattering volume, so this correlation is the reason of the differences of the Doppler lidar signal statistics from the Gaussian one.

Fig. 1 presents the results of calculating $K(r)$ as a function $t = 2k\tau\sqrt{2e/3}$ for different $d_{1/2}/l$. We can see that the statistics of the Doppler lidar signal is Gaussian ($NG = 1$) only in the limited case ($d_{1/2}/l \gg 1$). In other cases, the signal statistics of the Doppler lidar is non-Gaussian. Also it follows from Fig. 1 that the Doppler lidar signal statistics depends strongly on the state of atmospheric turbulence, which is characterized by the scale of turbulence l for the exponential model.

Fig. 2 shows the numerical simulation of scale turbulence profile using Yamada-Melor model for October 6, 1999 for the CASES-99 site [1, 3, 4]. If we assume that the size of $2d_{1/2} \sim 80$, then $d_{1/2}/l \leq 1$ and the Doppler lidar signal is non-Gaussian.

6. Central Limit Theorem and Doppler lidar signal

It is clear from Eq. (1) that the Doppler lidar signal is the sum of a large number ($N_p \gg 1$) random variables. To satisfy the central limit theorem we must assume that position of particles must be independently from each other in the scattering volume in term of statistics. We show that the statistical independence of the particles from each other does not hold in the turbulent atmosphere.

We calculate the correlation function of position of two particles using Eq. (2). These calculations lead us to the following equation

$$\langle R_m(t) R_n(t)\rangle = \langle R_m\rangle\cdot\langle R_n\rangle + \langle R_m\rangle\cdot u_r t + \langle R_n\rangle\cdot u_r t + u_r^2 t^2 + K(R_m - R_n)t^2 \quad (16)$$

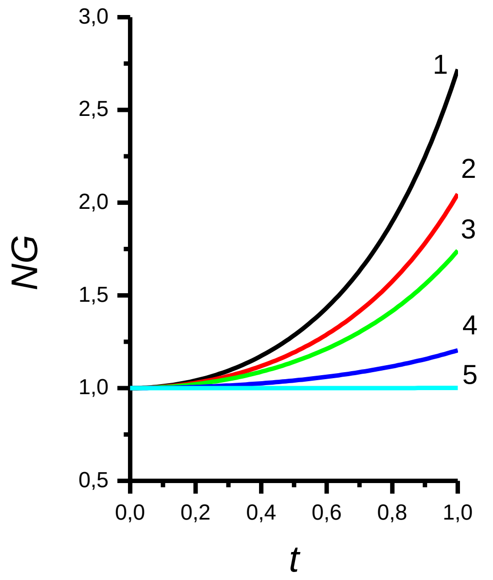

Fig. 1. NG vs. $t = 2k\tau\sqrt{2e/3}$.
Curve 1 – $d_{1/2}/l = 0$, 2 – $d_{1/2}/l = 0.5$,
3 – $d_{1/2}/l = 1$, 4 – $d_{1/2}/l = 5$, 5 – $d_{1/2}/l \gg 1$.

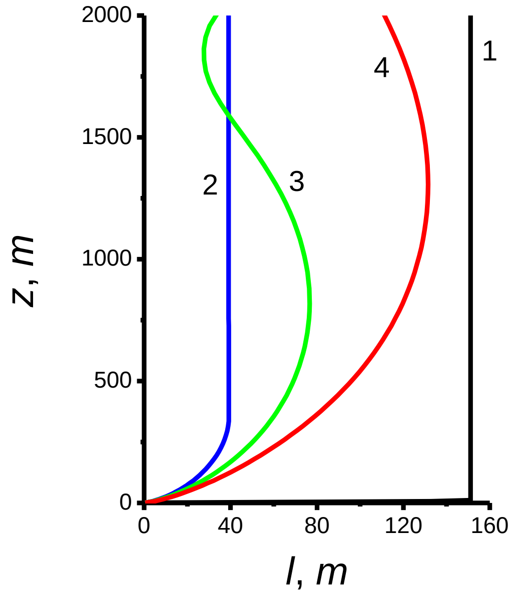

Fig. 2. Diurnal variations of turbulence scale profiles. Curve 1 – 6 a.m., 2 – 12 a.m., 3 – 3 p.m., 4 – 5 p.m.

It follows from Eq. (16) and $K(r) \neq 0$ that

$$\langle R_m(t) R_n(t) \rangle \neq \langle R_m(t) \rangle \cdot \langle R_n(t) \rangle$$

and statistical independence of the particles from each other does not hold in the turbulent atmosphere.
Thus, we cannot use the central limit theorem for the statistical analysis of the Doppler lidar signal in the turbulent atmosphere.

6. Conclusion

For coherent detection of optical fields scattered by a large number of particles ($N_p \gg 1$), the statistics of the Doppler lidar signal is non-Gaussian in the turbulent atmosphere. We cannot use the central limit theorem for the analysis of the signal statistics of the Doppler lidar. The main reason for the difference of the Doppler lidar signal statistics from the Gaussian is the fluctuations of wind velocity in the scattering volume.

7. References

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